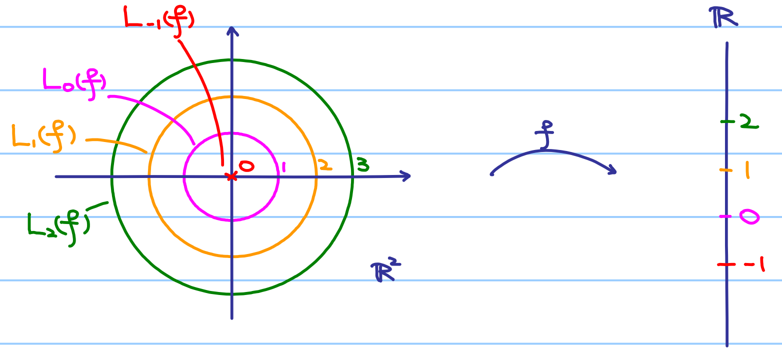
§15 Implicit Function Theorem and Inverse Function Theorem

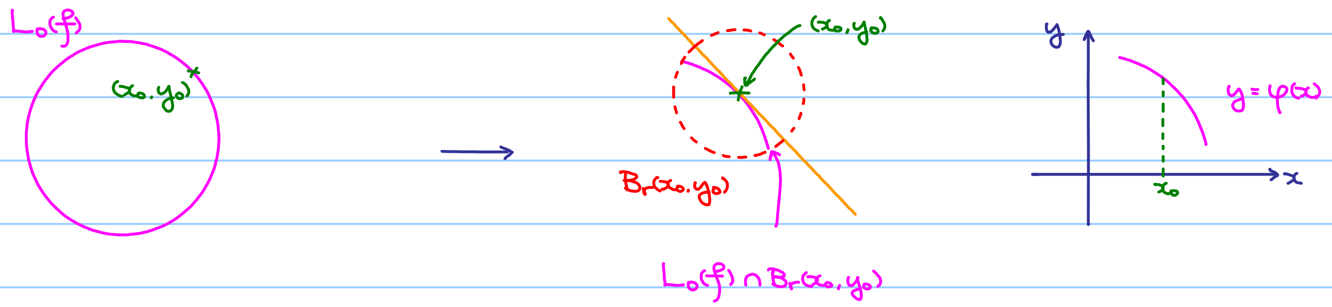
Implicit Function Theorem

Let $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^2+y^2-1$.



Then $L_0(f)$ is the unit circle centered at the origin.

Suppose that (x_0, y_0) lying on $L_0(f)$ such that $y_0 \neq 0$.



If $r>0$ is sufficiently small, $L_0(f) \cap B_r(x_0, y_0)$ is a small piece of arc of the circle, which is the graph of some function $y = \varphi(x) = \sqrt{1-x^2}$ such that

1) $\varphi(x_0) = y_0$
2) $f(x, \varphi(x)) = 0$

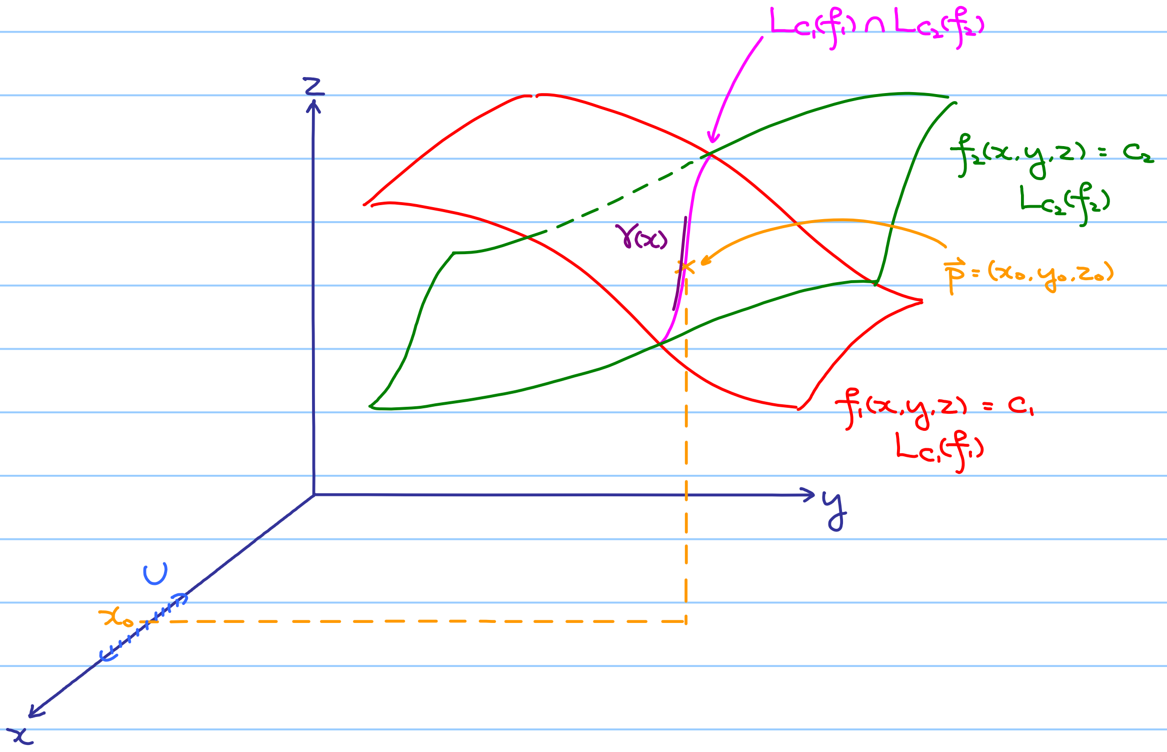
Idea: $L_0(f)$ is given by a relation $x^2+y^2-1=0$, but around (x_0, y_0), the relation is converted into function of x.

Idea:

Let $f : \mathbb{R}^{3=1+2} \to \mathbb{R}^2$ and $\vec{c} = (c_1, c_2) \in \mathbb{R}^2$. If we write $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z))$

$$f(x,y,z) = \vec{c} \iff \begin{cases} f_1(x,y,z) = c_1 \\ f_2(x,y,z) = c_2 \end{cases}$$

This equation gives relations between the variables x, y, z.



For any $(x,y,z) \in L_{c_1}(f_1) \cap L_{c_2}(f_2)$, we have $f(x,y,z) = \vec{c}$.

Fix $\vec{p} = (x_0, y_0, z_0) \in L_{c_1}(f_1) \cap L_{c_2}(f_2)$

Question: Can $L_{c_1}(f_1) \cap L_{c_2}(f_2)$ be parametrized locally around $\vec{p}$,

i.e. $L_{c_1}(f_1) \cap L_{c_2}(f_2)$ is given by $\gamma(x) = (x, \underbrace{y(x), z(x)}_{\varphi(x) = (y(x), z(x))})$ locally around $\vec{p}$?

$x_0 \in \underset{\text{open}}{U \subseteq} \mathbb{R}^1$, $\varphi : U \to \mathbb{R}^2$

① $\varphi(x_0) = (y_0, z_0)$

② $\gamma(x) = (x, \varphi(x)) \in L_{c_1}(f_1) \cap L_{c_2}(f_2)$, so $f(\gamma(x)) = f(x, \varphi(x)) = \vec{c}$

Answer : Implicit Function Theorem

Recall : $Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}$

If $\begin{bmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{bmatrix}$ is invertible at $\vec{p}$, i.e. $\det\left(\begin{bmatrix} \frac{\partial f_1}{\partial y}(\vec{p}) & \frac{\partial f_1}{\partial z}(\vec{p}) \\ \frac{\partial f_2}{\partial y}(\vec{p}) & \frac{\partial f_2}{\partial z}(\vec{p}) \end{bmatrix}\right)$ —— (*)

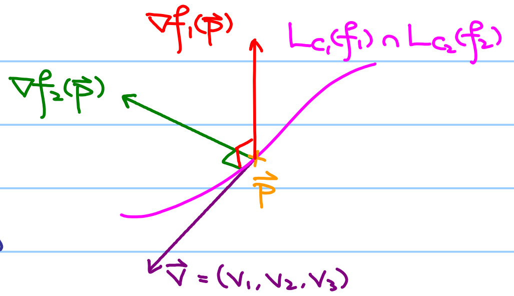
then yes !

Think : $\nabla f_1(\vec{p}) \perp L_{c_1}(f_1)$ at $\vec{p}$, $\nabla f_2(\vec{p}) \perp L_{c_2}(f_2)$ at $\vec{p}$

$\Rightarrow \nabla f_1(\vec{p}), \nabla f_2(\vec{p}) \perp L_{c_1}(f_1) \cap L_{c_2}(f_2)$ at $\vec{p}$

i.e. if $\vec{v}$ is a tangent vector of γ at $\vec{p}$, then $\nabla f_1(\vec{p}), \nabla f_2(\vec{p}) \perp \vec{v}$.

$$\begin{bmatrix} \frac{\partial f_1}{\partial x}(\vec{p}) & \frac{\partial f_1}{\partial y}(\vec{p}) & \frac{\partial f_1}{\partial z}(\vec{p}) \\ \frac{\partial f_2}{\partial x}(\vec{p}) & \frac{\partial f_2}{\partial y}(\vec{p}) & \frac{\partial f_2}{\partial z}(\vec{p}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$\left[\begin{array}{ccc|c} \frac{\partial f_1}{\partial x}(\vec{p}) & \frac{\partial f_1}{\partial y}(\vec{p}) & \frac{\partial f_1}{\partial z}(\vec{p}) & 0 \\ \frac{\partial f_2}{\partial x}(\vec{p}) & \frac{\partial f_2}{\partial y}(\vec{p}) & \frac{\partial f_2}{\partial z}(\vec{p}) & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \end{array}\right] \text{ by } (*)$$

$\therefore (v_1, v_2, v_3) = (t, -at, -bt)$ for $t \in \mathbb{R}$

$\uparrow$

If we move along γ, when we pass through $\vec{p}$, we do "move" in x direction,

so we can parametrize $L_{c_1}(f_1) \cap L_{c_2}(f_2)$ locally around $\vec{p}$ by x.

Theorem 15.1 (Implicit Function Theorem)

Let $D \subseteq \mathbb{R}^{n+k}$ be an open subset and let $f: D \to \mathbb{R}^k$ be a continuously differentiable function.

Points in D are expressed as $(\vec{x}, \vec{y}) = (x_1, \cdots, x_n, y_1, \cdots, y_k)$

If $f(\vec{x},\vec{y}) = (f_1(\vec{x},\vec{y}), f_2(\vec{x},\vec{y}), \cdots, f_k(\vec{x},\vec{y}))$, then

$$DF(\vec{x},\vec{y}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_k} \\ \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} & \frac{\partial f_k}{\partial y_1} & \cdots & \frac{\partial f_k}{\partial y_k} \end{bmatrix}$$

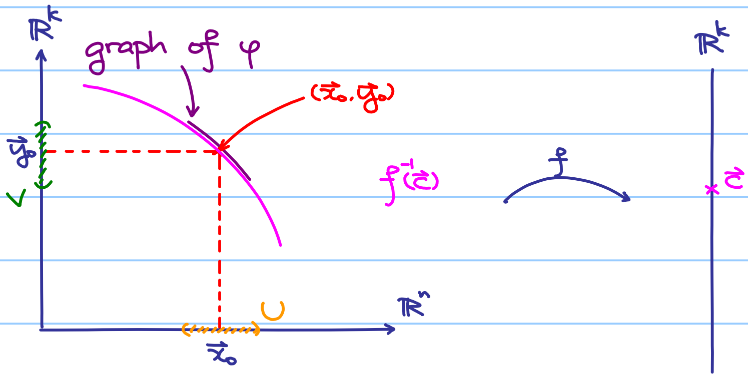
Let $\vec{p} = (\vec{x}_0, \vec{y}_0)$ If $f(\vec{p}) = f(\vec{x}_0, \vec{y}_0) = \vec{c}$ and $\begin{bmatrix} \frac{\partial f_1}{\partial y_1}(\vec{p}) & \cdots & \frac{\partial f_1}{\partial y_k}(\vec{p}) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial y_1}(\vec{p}) & \cdots & \frac{\partial f_k}{\partial y_k}(\vec{p}) \end{bmatrix}$ is invertible

(i.e. $\det\left(\frac{\partial f_i}{\partial y_j}(\vec{p})\right) \neq 0$), then there exist open subsets $U \subseteq \mathbb{R}^n$ containing $\vec{x}_0$ and $V \subseteq \mathbb{R}^k$ containing $\vec{y}_0$ and a unique continuously differentiable function $\varphi: U \to V$ such that

1) $\varphi(\vec{x}_0) = \vec{y}_0$

2) $f(\vec{x}, \varphi(\vec{x})) = \vec{c}$

3) $\left[\frac{\partial \varphi_i}{\partial x_\ell}\right]_{\substack{1 \le i \le k \\ 1 \le \ell \le n}} = -\left[\frac{\partial f_i}{\partial y_j}\right]^{-1}_{1 \le i,j \le k} \left[\frac{\partial f_j}{\partial x_\ell}\right]_{\substack{1 \le j \le k \\ 1 \le \ell \le n}}$ $\leftarrow$ gives relation between partial derivatives of f and φ



Idea: Around $(\vec{x}_0, \vec{y}_0)$, the relation $f(\vec{x},\vec{y}) = \vec{c}$ becomes $\vec{y} = \varphi(\vec{x})$.

Caution: If $\begin{bmatrix} \frac{\partial f_1}{\partial y_1}(\vec{p}) & \cdots & \frac{\partial f_1}{\partial y_k}(\vec{p}) \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial y_1}(\vec{p}) & \cdots & \frac{\partial f_k}{\partial y_k}(\vec{p}) \end{bmatrix}$ is not invertible, it is inconclusive.

(3) is actually a consequence of chain rule:

We have $y_i = \varphi_i(\vec{x}) = \varphi_i(x_1, x_2, \cdots, x_n)$ for $1 \le i \le k$.

$$f(\vec{x}, \varphi(\vec{x})) = \vec{c} = (c_1, c_2, \cdots, c_k) \Rightarrow f_i(\vec{x}, \varphi(\vec{x})) = c_i$$

$$\frac{\partial}{\partial x_\ell} f_i(\vec{x}, \varphi(\vec{x})) = \frac{\partial}{\partial x_\ell} c_i = 0$$

$$\frac{\partial f_i}{\partial x_\ell} + \sum_{j=1}^{k} \frac{\partial f_i}{\partial y_j} \frac{\partial y_j}{\partial x_\ell} = 0$$

$$\frac{\partial f_i}{\partial x_\ell} = -\sum_{j=1}^{k} \frac{\partial f_i}{\partial y_j} \frac{\partial y_j}{\partial x_\ell}$$

$$\therefore \left[\frac{\partial f_j}{\partial x_\ell}\right]_{\substack{1 \le j \le k \\ 1 \le \ell \le n}} = -\left[\frac{\partial f_i}{\partial y_j}\right]_{1 \le i, j \le k} \left[\frac{\partial \varphi_i}{\partial x_\ell}\right]_{\substack{1 \le i \le k \\ 1 \le \ell \le n}}$$

$$\left[\frac{\partial \varphi_i}{\partial x_\ell}(\vec{x})\right]_{\substack{1 \le i \le k \\ 1 \le \ell \le n}} = -\left[\frac{\partial f_i}{\partial y_j}(\vec{x}, \vec{y})\right]^{-1}_{1 \le i, j \le k} \left[\frac{\partial f_j}{\partial x_\ell}(\vec{x}, \vec{y})\right]_{\substack{1 \le j \le k \\ 1 \le \ell \le n}}$$

Example 15.1

Let $f(x,y) = x^2 + y^2 - 1$.

$Df(x,y) = \nabla f(x,y) = [2x \quad 2y]$

Suppose that $f(x_0, y_0) = x_0^2 + y_0^2 - 1 = 0$.

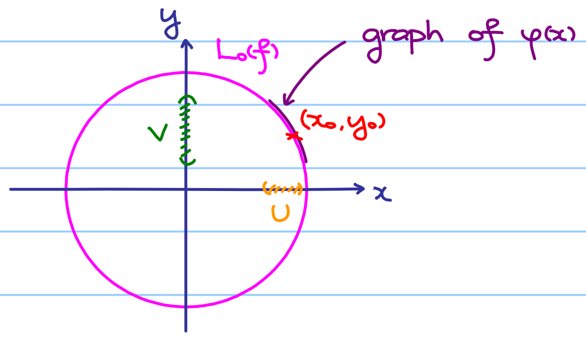
$\nabla f(x_0, y_0) = [2x_0 \quad 2y_0]$

If $y_0 \ne 0$, $[2y_0]$ is invertible,

so by implicit function theorem,

around (x_0, y_0), $L_0(f)$ can be expressed

as graph of function $\varphi: U \to V$ $(x \mapsto y)$ with $x_0 \mapsto y_0$.



However, if $y_0 = 0$, $x_0 = \pm 1$, $[2y_0]$ is not invertible and around $(x_0, y_0) = (\pm 1, 0)$,

$L_0(f)$ cannot be expressed as graph of a function of x.

However, if we reorder the coordinates

and write $f(y,x) = x^2 + y^2 - 1$,

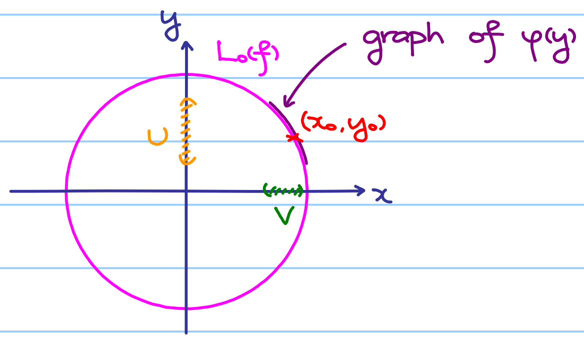
then $\nabla f(y_0, x_0) = [2y_0 \quad 2x_0]$.

If $x_0 \ne 0$, $[2x_0]$ is invertible,

so by implicit function theorem,

around (x_0, y_0), $L_0(f)$ can be expressed

as graph of function $\varphi: U \to V$ $(y \mapsto x)$ with $y_0 \mapsto x_0$.



Example 15.2

Let $f(x,y) = y^3 - \sin^3 x$

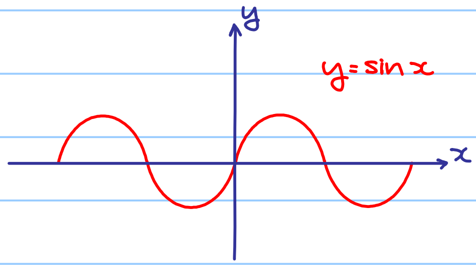
Consider $f(x,y) = y^3 - \sin^3 x = 0$

$$(y-\sin x)(y^2 + y\sin x + \sin^2 x) = 0$$

$$y = \sin x \quad \text{or} \quad y^2 + y\sin x + \sin^2 x = 0$$

$$(\text{Ex}: \ y = \sin x = 0)$$

$$\therefore \ y = \sin x.$$



$L_0(f)$ is just the graph of $y = \varphi(x) = \sin x$.

Note that $f(0,0) = 0$, so $(0,0) \in L_0(f)$ and in particular, around $(0,0)$ $L_0(f)$ can be expressed as graph of a function of x.

However, $Df(x,y) = \nabla f(x,y) = \begin{bmatrix} 3\sin^2 x \cos x & 3y^2 \end{bmatrix}$ and $\nabla f(0,0) = \begin{bmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$\left[\frac{\partial f}{\partial y}(0,0)\right] = [0]$ is not invertible.

Example 15.3

Consider the constraints

$$\begin{cases} xz + \sin(xy - z^2) = 8 \\ 3x + 4y + z = 18 \end{cases}$$

Around the point $(4,1,2)$, can we solve two of the variables as functions of the remaining variable ?

Let $f: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z))$

$$= (xz + \sin(xy - z^2), \ 3x + 4y + z)$$

Then, $Df(x,y,z) = \begin{bmatrix} z + y\cos(xy - z^2) & x\cos(xy - z^2) & x - 2z\cos(xy - z^2) \\ 3 & 4 & 1 \end{bmatrix}$

and $Df(4,1,2) = \begin{bmatrix} \frac{\partial f_1}{\partial x}(4,1,2) & \frac{\partial f_1}{\partial y}(4,1,2) & \frac{\partial f_1}{\partial z}(4,1,2) \\ \frac{\partial f_2}{\partial x}(4,1,2) & \frac{\partial f_2}{\partial y}(4,1,2) & \frac{\partial f_2}{\partial z}(4,1,2) \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ 3 & 4 & 1 \end{bmatrix}$.

$\begin{vmatrix} \frac{\partial f_1}{\partial y}(4,1,2) & \frac{\partial f_1}{\partial z}(4,1,2) \\ \frac{\partial f_2}{\partial y}(4,1,2) & \frac{\partial f_2}{\partial z}(4,1,2) \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 4 & 1 \end{vmatrix} = 4 \neq 0 \Rightarrow$ y, z can be expressed as functions of x.

$$\begin{vmatrix} \frac{\partial f_1}{\partial x}(4,1,2) & \frac{\partial f_1}{\partial z}(4,1,2) \\ \frac{\partial f_2}{\partial x}(4,1,2) & \frac{\partial f_2}{\partial z}(4,1,2) \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 3 & 1 \end{vmatrix} = 3 \neq 0 \Rightarrow$$ x, z can be expressed as functions of y.

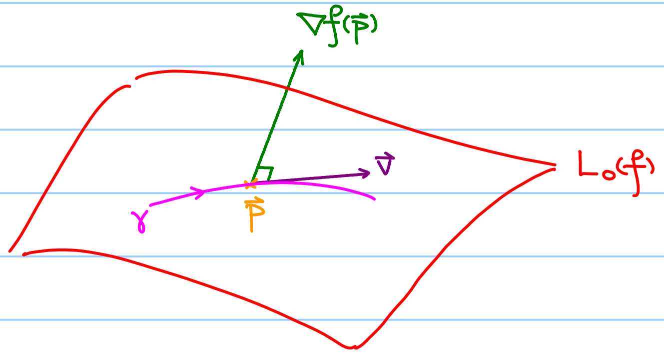
$$\begin{vmatrix} \frac{\partial f_1}{\partial x}(4,1,2) & \frac{\partial f_1}{\partial y}(4,1,2) \\ \frac{\partial f_2}{\partial x}(4,1,2) & \frac{\partial f_2}{\partial y}(4,1,2) \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 3 & 4 \end{vmatrix} = 0$$ No conclusion

Example 15.4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function.

Suppose that $\vec{p} \in L_0(f)$ such that $\nabla f(\vec{p}) \neq \vec{0}$.

Show that for all $\vec{v} \perp \nabla f(\vec{p})$, there exists a curve $\gamma : (-\delta, \delta) \to L_0(f)$ such that $\gamma(0) = \vec{p}$ and $\gamma'(0) = \vec{v}$.



We write $\vec{p} = (p_1, \cdots, p_n)$

Note $\nabla f(\vec{p}) \neq \vec{0}$,

so $\frac{\partial f}{\partial x_j}(\vec{p}) \neq 0$ for some $1 \le j \le n$.

Without loss of generality, assume $\frac{\partial f}{\partial x_n}(\vec{p}) \neq 0$.

By implicit function theorem,

there exist open subsets $U \subseteq \mathbb{R}^{n-1}$ containing $(p_1, \cdots, p_{n-1})$, $V \subseteq \mathbb{R}$ containing p_n and a unique continuously differentiable function $\varphi : U \to V$ such that

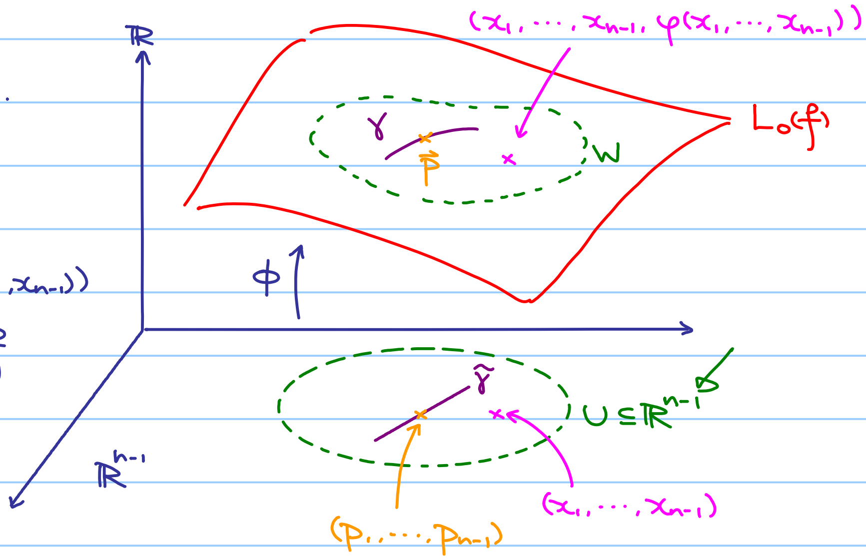
1) $\varphi(p_1, \cdots, p_{n-1}) = p_n$

2) $f(x_1, \cdots, x_{n-1}, \varphi(x_1, \cdots, x_{n-1})) = 0$.

Then, $\phi : U \to \mathbb{R}^n$ defined by

$\phi(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{n-1}, \varphi(x_1, \cdots, x_{n-1}))$

gives a local parametrization of $L_0(f)$ around $\vec{p}$.



(Remark: A point on $\phi(U) \subseteq L_0(f)$ correspond to exactly one $(x_1, \cdots, x_{n-1}) \in U$.)

Let $\vec{v}=(v_1,\cdots,v_n)\in\mathbb{R}^n$ such that $\vec{v}\perp\nabla f(\vec{p})$. Then,

$$\vec{v}\cdot\nabla f(\vec{p}) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(\vec{p}) = 0$$

$$\therefore\ v_n = -\sum_{i=1}^{n-1} v_i \frac{\partial f}{\partial x_i}(\vec{p}) \Big/ \frac{\partial f}{\partial x_n}(\vec{p}) \quad \text{—— } (*)$$

Define $\tilde{\gamma}:(-\delta,\delta)\to U$ by $\tilde{\gamma}(t)=(p_1,\cdots,p_{n-1})+t(v_1,\cdots,v_{n-1})$

(δ can be choosen such that $\tilde{\gamma}$ is contained in U)

and define $\gamma:(-\delta,\delta)\to\mathbb{R}^n$ by $\gamma(t)=(\tilde{\gamma}(t),\varphi(\tilde{\gamma}(t)))$.

Note : $f(x_1,\cdots,x_{n-1},\varphi(x_1,\cdots,x_{n-1}))=0$.

$$\frac{\partial}{\partial x_i} f(x_1,\cdots,x_{n-1},\varphi(x_1,\cdots,x_{n-1})) = \frac{\partial}{\partial x_i}0 = 0 \quad \text{for } i=1,2,\cdots,n-1$$

$$\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_n}\frac{\partial \varphi}{\partial x_i} = 0$$

$$\frac{\partial \varphi}{\partial x_i} = -\frac{\partial f}{\partial x_i} \Big/ \frac{\partial f}{\partial x_n}$$

$$\frac{\partial \varphi}{\partial x_i}(\tilde{\gamma}(0)) = -\frac{\partial f}{\partial x_i}(\gamma(0)) \Big/ \frac{\partial f}{\partial x_n}(\gamma(0)) = -\frac{\partial f}{\partial x_i}(\vec{p}) \Big/ \frac{\partial f}{\partial x_n}(\vec{p}) \quad \text{— } (**)$$

Then, $\gamma'(t) = (\tilde{\gamma}'(t), \nabla\varphi(\tilde{\gamma}(t))\cdot\tilde{\gamma}'(t))$

$$\gamma'(0) = (\tilde{\gamma}'(0), \nabla\varphi(\tilde{\gamma}(0))\cdot\tilde{\gamma}'(0))$$

$$= \Big(v_1,\cdots,v_{n-1}, \sum_{i=1}^{n-1}\frac{\partial\varphi}{\partial x_i}(\tilde{\gamma}(0))\cdot\tilde{\gamma}'(0)\Big)$$

$$= \Big(v_1,\cdots,v_{n-1}, -\sum_{i=1}^{n-1} v_i\frac{\partial f}{\partial x_i}(\vec{p}) \Big/ \frac{\partial f}{\partial x_n}(\vec{p})\Big) \quad \text{by } (**)$$

$$= (v_1,\cdots,v_{n-1}, v_n) \quad \text{by } (*)$$

$$= \vec{v}$$

Inverse Function Theorem

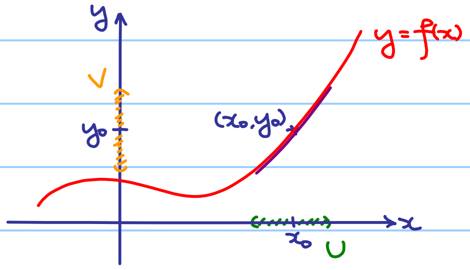
Let $f: \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function.

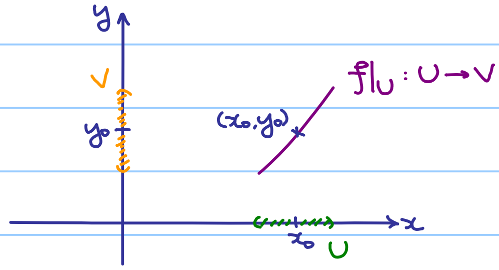
Suppose that $f(x_0) = y_0$ and $f'(x_0) > 0$ (if fact, we only need $f'(x_0) \neq 0$).

Since $f'(x_0) > 0$ and f' is continuous, $f'(x) > 0$ in a small neighborhood U of x_0.

Then, we know f is strictly increasing on U and so we should be able to construct a local inverse of f, i.e inverse function of $f|_U$.

Remark: $f|_U : U \to \mathbb{R}$ is the restriction of f on U, i.e. $f|_U(x) = f(x)$ for all $x \in U$.





Construct the inverse function $g = (f|_U)^{-1} : V \to U$

Furthermore, if we write $y = f(x) = f|_U(x)$ for $x \in U$, then $x = g(y)$ and we have (i) $g(f(x)) = x$ for all $x \in U$

(ii) $f(g(y)) = y$ for all $y \in V$

Consider (i) and differentiate both sides with respect to x,

$$\frac{d}{dx} g(f(x)) = \frac{d}{dx} x$$

$$g'(f(x)) \cdot f'(x) = 1 \qquad \text{(Chain rule)}$$

$$g'(f(x)) = \frac{1}{f'(x)}$$

$$\text{or} \quad \frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

In fact, the above result can be generalized as the following:

Theorem 15.2 (Inverse Function Theorem)

Let $D \subseteq \mathbb{R}^n$ be an open subset and let $f: D \to \mathbb{R}^n$ be a continuously differentiable function.

Suppose that $f(\vec{x}_0) = \vec{y}_0$ and $Df(\vec{x}_0) \in M_n(\mathbb{R})$ is invertible.

Then, there exist open subsets $U \subseteq \mathbb{R}^n$ containing $\vec{x}_0$ and $V \subseteq \mathbb{R}^n$ containing $\vec{y}_0$ and a continuously differentiable function $g: V \to U$ such that

1) $g(\vec{y}_0) = \vec{x}_0$

2) $g(f(\vec{x})) = \vec{x}$ for all $\vec{x} \in U$
   $f(g(\vec{y})) = \vec{y}$ for all $\vec{y} \in V$ $\Big\}$ g is a local inverse of f, i.e. $g = (f|_U)^{-1}$

3) $Dg(f(\vec{x})) = Df(\vec{x})^{-1}$ for all $\vec{x} \in U$

Caution: If $Df(\vec{x}_0) \in M_n(\mathbb{R})$ is not invertible, it is inconclusive

Example 15.4

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (x^2 - y^2, 2xy)$.

Clearly, $f(-x,-y) = f(x,y)$ and so f is not injective and it has no global inverse.

How about local inverse?

Note that $Df(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$ and $\det(Df(x,y)) = 4(x^2+y^2) \geq 0$.

Therefore, $Df(x,y)$ is invertible for all $(x,y) \neq (0,0)$.

Example 15.5

Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

We have $f'(0) = 0$ and local inverse of f at $x=0$ does not exist.

Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$

We have $f'(0) = 0$ and local inverse of f at $x=0$ exist.

(In fact, $f^{-1}(x) = \sqrt[3]{x}$ is a global inverse.)

Example 15.6

Suppose that $\begin{cases} u = xy\sin\frac{\pi x}{2} \\ v = x^2 y\cos\pi y \end{cases}$. When $(x,y) = (1,1)$, we have $(u,v) = (1,-1)$

If u and v are increased by 0.1, how do x and y change?

Let $(u,v) = f(x,y)$. Note that $Df(1,1) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ which is invertible.

Therefore, inverse of f, denoted by g, exists locally around $(x,y) = (1,1)$.
We have $(x,y) = g(u,v)$ and note that $\begin{cases} \Delta x \approx \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v, \\ \Delta y \approx \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v \end{cases}$

in other words, $\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = Dg \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}$.

Therefore, $\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = Dg(1,-1)\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} = Df(1,1)^{-1}\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.2 \\ 0.3 \end{bmatrix}$$

Remark: Even inverse of f exists locally around $(x,y) = (1,1)$, it is difficult to write down explicitly.